Section 5.1

Approximating Net Change

If an object moves at a constant velocity of $v = 4$ m/s for 5 seconds, how far has the object moved?

If the objects velocity changes over time, say $v(t) = t^2$, how far has the object moved in 5 seconds?

Compute a better approximation:

Definition 5.1.1. Suppose $[a, b]$ is a closed interval containing n subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_n - 1, x_n]$$

of equal length $\Delta x = x_i - x_{i-1} = \dfrac{b-a}{n}$, with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, \ldots, x_n$ are called *grid points*, and the create a *regular partition* of $[a, b]$.

The formula for the k^{th} grid point is

$$x_k =$$

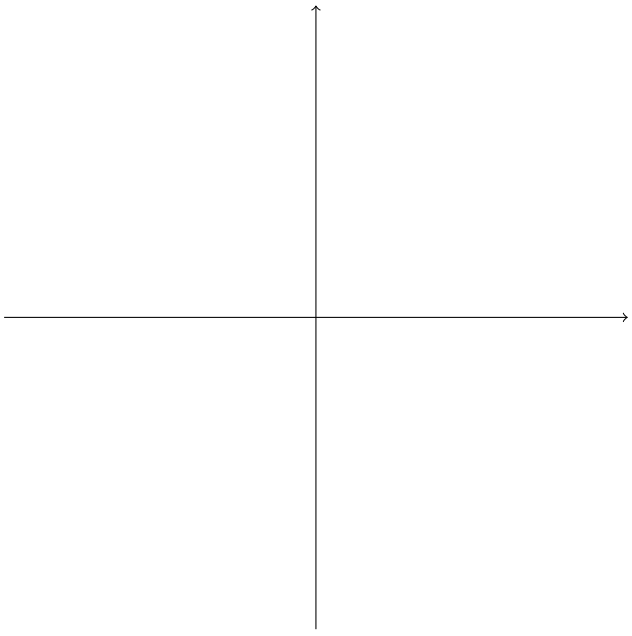
Definition 5.1.2. From each subinterval $[x_{i-1}, x_i]$ of a regular partition on $[a, b]$ we choose a *sample point* x_i^*. The *Riemann sum* of f is

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

approximates the net change $F(b) - F(a)$, where $F'(x) = f(x)$.

Example 5.1.3. A winch located atop a 30 m building is pulling a steal cable from the ground weighing 4 N/m. If the cable reaches from the winch to the ground, approximate the amount of work that is needed to pull the cable completely up.

Visualize:



There are certain Riemann sums that are particularly useful:

Definition 5.1.4. Suppose f is defined on a closed interval $[a, b]$, which is divided into a regular partition with n subintervals. The Riemann sum

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a

1. *left Riemann sum* if x_i^* is the left endpoint of $[x_{i-1}, x_i]$.
2. *right Riemann sum* if x_i^* is the right endpoint of $[x_{i-1}, x_i]$.
3. *midpoint Riemann sum* if x_i^* is the midpoint of $[x_{i-1}, x_i]$.

for each $k = 1, 2, \ldots, n$.

Example 5.1.5. Let R be the region bounded by $f(x) = 3\sqrt{x}$ and the x-axis from $x = 4$ to $x = 16$. Approximate the area of R using a left Riemann sum with $n = 6$ subintervals.

Approximate the area of R with a right Riemann sum with $n = 6$ subintervals.

Summation Notation

To make writing long sums less cumbersome we have a special notation: the sum of the numbers $a_k = f(k)$ for k ranging from 1 to n is written

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n.$$

The k is a *dummy variable*, meaning it is internal to the sum, can be any letter we like, and is just used for counting. N.B. we can make k start at other values beside 1.

Example 5.1.6. Compute

1. $\sum_{k=1}^{5} k^2$

2. $\sum_{k=0}^{4} (2k+1)$

3. $\sum_{j=1}^{100} j$

Theorem 5.1.7 (Sum Rules)**. Constant Multiple Rule:** $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

Addition Rule: $\sum_{k=1}^{n} \big(a_k + b_k\big) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Translation Rule: $\sum_{k=m}^{n} a_k = \sum_{k=M}^{n+M-m} a_{k-M+m}$

Example 5.1.8. $\sum_{k=0}^{4}(2k+1)$

Theorem 5.1.9 (Useful Sum Formulas)**.** Let n be a positive integer and c a real number.

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Let's finish by writing out left, right, and midpoint Riemann sums for f on $[a, b]$ in sigma notation: